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EFFECT OF BULK COMPRESSIBILITY ON THE STIFFNESS OF CYLINDRICAL BASE ISOLATION BEARINGS

MICHEL S. CHALHOUB
The Ralph M. Parsons Company, 100 West Walnut Street, Pasadena, CA 91124, U.S.A.

and

JAMES M. KELLY
Civil Engineering Department, University of California, Berkeley, CA 94720, U.S.A.

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Abstract—The seismic design technique based on mounting building structures on horizontally flexible foundations is becoming popular. The horizontal flexibility accompanied by a very high vertical stiffness is well realized by multilayered elastomeric bearings made of interleaved steel plates and rubber slices. The rubber is vulcanized to the steel to ensure bond. Several design expressions for such pads were already proposed. However, these expressions either consider the rubber incompressible or they account for compressibility by an *ad hoc* modified formula. In this paper, the governing equations for the pressure in a rubber slice of arbitrary cross section are presented. The solution is carried out for the circular shape and compared to experimental results. It was found that the formula for the compressive stiffness of those bearings developed here by including the effect of bulk compressibility predicts very accurately the measured stiffness. The classically used formula that ignores the rubber compressibility yielded exaggerated over-estimation of the compression modulus. The *ad hoc* modified formula gave closer results but was still inaccurate. An expression derived from the exact solution and a simplified form are herein provided to replace the *ad hoc* modified formula and the expression that ignores rubber compressibility.

NOTATION

A	area of confined rubber layer
A_0	area of rubber bearing
$(EI)_1$	effective bending stiffness including effect of bulk compressibility
$(EI)_1^\infty$	effective bending stiffness excluding effect of bulk compressibility
E_1	effective compression modulus including effect of bulk compressibility
E_0	Young's modulus for rubber
E_1^∞	effective compression modulus excluding effect of bulk compressibility
E'_1	sum of E_0 and E_1^∞
G	shear modulus for rubber
K	bulk compression modulus
K_H	horizontal stiffness of rubber bearing
K_V	vertical stiffness of rubber bearing
M	bending moment
p	pressure in rubber
P	total axial force resisted by rubber layer
P_1	additional axial force resisted due to kinematic constraint
P_0	axial force resisted by lubricated rubber layer
P_{ave}	average axial load in cyclic compression tests
R	radius of confined rubber layer
R_0	radius of rubber bearing
S	shape factor of confined rubber layer
t	thickness of rubber layer
t_h	total thickness of rubber in bearing
α	angle of tilt
β	constant related to hardness of rubber
ε_{ii}	linear strain in the i direction ($i = x, y, z$)
ε_c	compressive strain in rubber layer
λ^2	constant equal to $12G/Kt^2$
ρ	radius of curvature.

INTRODUCTION

In recent years interest has increased in the use of rubber isolators for the earthquake protection of buildings and other structures. These isolators are very similar to thermal expansion bearings for highway bridges but they differ in that they may have to accept large lateral displacements and there is often a separate requirement of very high vertical stiffness. To accommodate the conflicting demands of the large lateral displacement and high vertical stiffness it is often necessary to resort to bearings with many thin layers. Each layer will have a shape factor S, defined as the ratio of the loaded area to the load free area, in the range of 10 to 30. For such high shape factor pads the usual assumption of incompressibility in the rubber material no longer holds, and it becomes necessary to modify the standard theory for the effective compression and effective tilting stiffness to account for the bulk compressibility of the rubber. In this paper, the appropriate modifications of the theory are developed and design rules for high shape factor pads are given. The closed form solution developed here involves Bessel functions which were expanded in terms of their arguments to yield expressions that are simpler to use. Each one of these simplified equations is recommended for a certain range of shape factor.

Half-scale bearings were tested at the Earthquake Engineering Research Center of the University of California at Berkeley. The vertical and horizontal stiffnesses were measured for various axial loads and levels of shear strain. The experimental results from tests in pure compression were compared to the theoretically determined values and excellent agreement was found. Hence, the final expressions presented here are simple and accurate and it is hoped that they will be adopted by the engineering profession.

Even though the governing equations presented here apply to a rubber slice of any shape, focus will be on circular cross sections. A theoretical solution and discussion about annular and square cross sectional areas are presented by Chalhoub and Kelly (1986).

BASIC ASSUMPTIONS AND GOVERNING EQUATIONS

The compressive stiffness of a rubber slice confined between two rigid steel plates depends on the level of steel–rubber bond. Furthermore when perfect bond is assumed, there is a significant difference in the effective compressive stiffness of the pad whether the rubber is assumed incompressible or compressible.

Starting from certain kinematic assumptions, the equations governing the hydrostatic pressure in the rubber are developed for a thin slice constrained between two steel plates. Two different cases are considered. When the steel–rubber contacting surfaces are perfectly lubricated, the rubber is free to move horizontally when the unit is subjected to vertical strain ε_c. The compression modulus in this case is simply the Young's modulus of the elastomer, E_0, and the vertical load resisted under this deformation is $P_0 = E_0 A \varepsilon_c$. If the rubber is perfectly bonded to the steel plates, an additional stiffness is developed due to the kinematic constraint. In this case, it is assumed that under direct compression horizontal planes remain plane and horizontal, and that a vertical straight line deforms into a parabola. This is a commonly accepted theory for bonded rubber blocks, and a similar approach was presented by Gent and Meinecke (1970) and by Stanton and Roeder (1982). Within this second case where the rubber is perfectly bonded to the steel, the equations for the hydrostatic pressure are developed for the case where the elastomer is considered incompressible and compressible, respectively. The additional stiffness, which will be adopted as the effective compression modulus will be denoted by E_1^{∞} or by E_1 whether the rubber is considered incompressible or compressible respectively.

The governing equations for the pressure in the rubber, namely eqns (1), (2), (3) and (4) are presented here while their derivation is presented in the Appendix. Consider a single layer of rubber of thickness t, sandwiched between two rigid steel plates. The layer, the reference axes $(0x, y, z)$, and the two deformation patterns discussed below are shown in Fig. 1. When change in volume is neglected, the equation which is solved for p is in essence an integration of the equation of incompressibility, namely

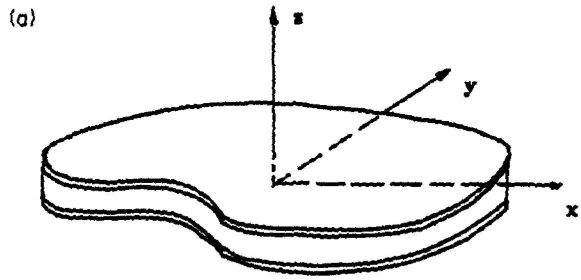

Fig. 1a. Rubber layer between rigid steel plates.

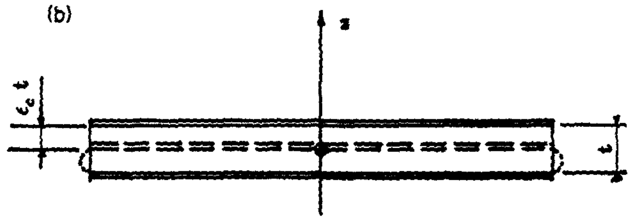

Fig. 1b. Uniformly compressed rubber layer.

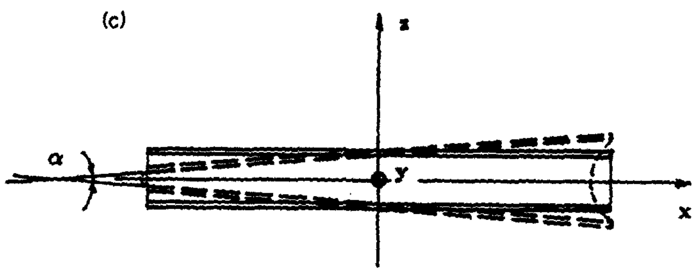

Fig. 1c. Rubber layer in pure flexure.

$$\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}=0$$

the integration being carried out through the thickness of the pad. This leads to a basic equation for $p(x, y)$ for the case of pure compression in the form

$$\nabla^2 p+\frac{12G\varepsilon_c}{t^2}=0 \tag{1}$$

over the area of the pad with zero pressure on its boundary. The solution of this equation is used to calculate the total axial load P_1 by integration of the pressure p over the cross sectional area A.

When bulk compression is included, eqn (1) is readily modified by noting that the change in volume is given by $-p/K$, where K is the bulk modulus. Replacing the incompressibility constraint equation by

$$\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}=-p/K$$

and on integration through the thickness this leads to an equation in the form

$$\nabla^2 p - \lambda^2 (p - K\varepsilon_c) = 0 \tag{2}$$

where $\lambda^2 = 12G/Kt^2$, with the boundary condition of zero pressure on the free edges of the pad.

In eqns (1) and (2) the pressure p refers to the additional term produced by the constraint caused by the steel reinforcement. Denoting the additional force by $P_1 = \int_A p \, dA$, the total vertical resultant is $P = P_0 + P_1$. It should be noted that the effect of compressibility is only important in high shape factor pads and in this case the term $E_0 A \varepsilon_c$ which is unchanged with the shape factor S, is negligible compared to the additional stiffness produced by the kinematic constraint. Thus, the actual compression modulus is determined by $E'_1 = P/(A\varepsilon_c) = E_0 + E_1 \doteq E_1$ (or E_1^∞).

When change in volume is neglected, the basic equation for the derivation of the tilting stiffness, $(EI)_1^\infty$, is

$$\nabla^2 p - \frac{12G}{t^2}\frac{x}{\rho} = 0 \tag{3}$$

where ρ is the curvature t/α, and α is the total tilt angle for a single layer (Fig. 1c). When similarly modified to account for bulk compression it becomes

$$\nabla^2 p - \lambda^2 (p + Kx/\rho) = 0. \tag{4}$$

The process by which the tilting stiffness $(EI)_1^\infty$ is determined for K infinite and K finite, is first to evaluate the pressure p in eqns (3) and (4) respectively, with appropriate boundary conditions and determine the total moment M, from $\int_A px \, dA$. The stiffness $(EI)_1^\infty$ is then given by ρM.

In the following sections, we will solve the preceding equations for the pressure distributions under pure compression and pure moment for pads of circular shape, and determine the compression and tilting stiffnesses. These equations will be expressed and solved in polar coordinates.

COMPRESSION EXCLUDING VOLUME CHANGE

Consider a circular slice of rubber of thickness t and radius R (Fig. 2), compressed between two steel plates. For pure compression, the stress state is axisymmetric, $p(r, \theta) = p(r)$ and eqn (1) becomes

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{12G\varepsilon_c}{t^2} = 0 \tag{5}$$

with the boundary condition of zero pressure on the circle delimiting the section. Expressing $p(r)$ in the form $C_1 r^2 + C_2 r + C_3$, where C_1, C_2 and C_3 are arbitrary constants, and substituting in eqn (5) leads to $4C_1 + C_2/r + 12G\varepsilon_c/t^2 = 0$. Since a bounded solution is expected, $C_2 \equiv 0$ and thus $C_1 = -3G\varepsilon_c/t^2$. Applying the condition $p(R) = 0$ yields C_3, and

$$p(r) = 3G\varepsilon_c (R^2 - r^2)/t^2. \tag{6}$$

The compressive load resisted under this pressure distribution is

$$P_1 = \int_A p(r) \, dr = \frac{3}{2}\frac{\pi G \varepsilon_c}{t^2} R^4$$

and the total vertical load is then

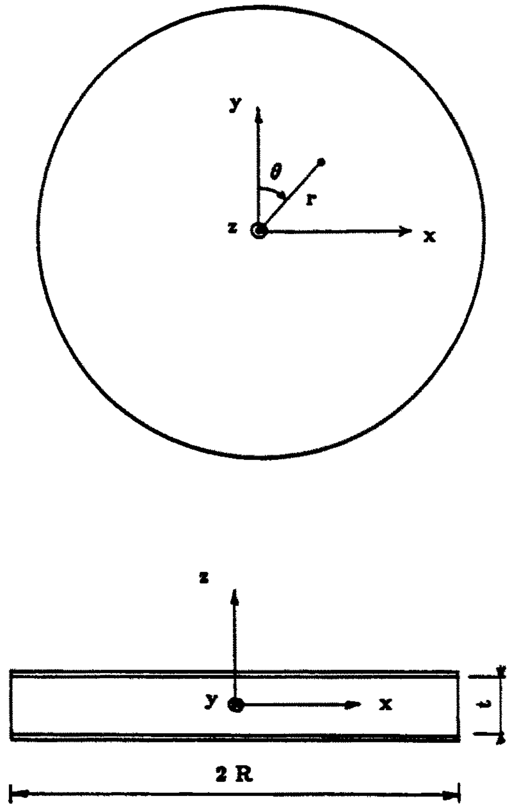

Fig. 2. Circular rubber layer.

$$P = E_0 A\varepsilon_c + \frac{3}{2}\frac{\pi G\varepsilon_c}{t^2}R^4 = E'_1 A\varepsilon_c. \tag{7}$$

Noting that the shape factor, defined by the ratio of the loaded area to the load-free area, is $S = R/2t$ for a circular slice, and that the cross sectional area is $A = \pi R^2$, eqn (7) can be written as $P = A(E_0 + 6GS^2)\varepsilon_c$, which provides the expression for the effective compression stiffness, namely $E'_1 = E_0 + 6GS^2$. Since incompressibility is assumed, $G = E_0/3$ and

$$E'_1 = E_0(1 + 2S^2). \tag{8}$$

A similar result is presented by Gent and Lindley (1959) and by Lindley (1979) for circular rubber blocks and by Chalhoub and Kelly (1988) for long rectangular cross sections. However, for long rectangles, the additional factor was found to be $4S^2/3$ instead of $2S^2$. Also, the semi-empirical relation $E'_1 = E_0(1 + 2\beta S^2)$ is presented by Stanton and Roeder (1982) and by Allen *et al.* (1966). In the preceding expression β is a factor less than one, determined empirically, and varies with the hardness of the material used. The hardness is one of the characteristics of vulcanized rubber considered in the design of bearings. It is related to Young's modulus since it is measured by an elastic penetration test using a specially-shaped indentor. A commonly used scale is the International Rubber Hardness Degree (IRHD). Experimental values of β in relation to other elastic properties are provided by Allen *et al.* (1966).

COMPRESSION INCLUDING VOLUME CHANGE

When bulk compressibility is included, we must solve eqn (2) with the condition of zero pressure on the circle delimiting the section

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}-\lambda^2(p-K\varepsilon_c)=0. \tag{9}$$

$K\varepsilon_c$ is a particular solution of the above equation. The homogeneous equation is a modified Bessel equation of order zero and parameter λ

$$r^2p_{,rr}+rp_{,r}-\lambda^2r^2p=0$$

and has the solution $C_4I_0(\lambda r)+C_5K_0(\lambda r)$, where I_0 and K_0 are the modified Bessel functions of first and second kind respectively, and of order zero. C_4 and C_5 are arbitrary constants. Since the solution is expected to be bounded at the origin, $C_5\equiv 0$. The boundary condition $p(R)=0$ provides C_4, and the solution is

$$p(r)=K\left[1-\frac{I_0(\lambda r)}{I_0(\lambda R)}\right]\varepsilon_c. \tag{10}$$

By integration over the area for the vertical load resisted under this pressure distribution

$$\begin{aligned}P_1&=2\pi K\varepsilon_c\int_0^R(1-I_0(\lambda r)/I_0(\lambda R))r\,\mathrm{d}r\\&=\frac{2KA}{(\lambda R)^2}((\lambda R)^2/2-\lambda RI_1(\lambda R)/I_0(\lambda R))\varepsilon_c.\end{aligned}$$

From the above expression, the effective compression stiffness is

$$E_1=K\left[1-\frac{2}{\lambda R}\frac{I_1(\lambda R)}{I_0(\lambda R)}\right] \tag{11}$$

where I_1 is the modified Bessel function of first kind, of order one. In order to compare the present result to the expression obtained in the case of incompressibility, the Bessel functions are developed for small arguments and the first few terms are retained

$$\begin{aligned}I_0(x)&=1+\frac{x^2}{4}+\frac{x^4}{64}+\frac{x^6}{2304}+\cdots\\I_1(x)&=\frac{x}{2}+\frac{x^3}{16}+\frac{x^5}{384}+\cdots.\end{aligned} \tag{12}$$

Setting the ratio $I_1(x)/I_0(x)$ and using the binomial formula, we have

$$\frac{I_1(x)}{I_0(x)}=x^2/8-x^4/48+0(x^6)$$

and with $x=\lambda R$

$$I_1(\lambda R)/I_0(\lambda R)=(\lambda R)^2/8-(\lambda R)^4/48.$$

Substituting in the expression for the stiffness and replacing λ by its value leads to

$$E_1=6GS^2(1-8GS^2/K). \tag{13}$$

Note that the effective compressive stiffness corresponding to the case where incompressibility was assumed (previous section) is recovered, but here it is reduced by a factor of $8GS^2/K$ because of the effect of bulk compression. Also, when K approaches infinity the reduction term approaches zero.

Equation (13) has the limitation that it can be used only for shape factors up to a certain value since it was obtained from an approximation of the Bessel functions by their series for small arguments. For instance, imposing an error of about 2% on the expression of E_1 due to the truncation in the expressions of $I_0(x)$ and $I_1(x)$, we conclude that eqn (13) can be used for $S \leqslant (K/12G)^{1/2}$. For commonly used rubber compounds in this type of bearings, G varies between 0.9 MPa and 1.4 MPa and K is around 2070 MPa. Taking the high value of G in the denominator to obtain the limiting condition on S, leads to $S \leqslant 11$. However, eqn (13) served to show explicitly the reduction factor in the expression of E'_1 when compressibility of the material is included. In order to compare eqn (13) to an expression previously proposed by Gent and Meinecke (1970), it can be rearranged for small values of $8GS^2/K$ to read

$$\frac{1}{E_1} = \frac{1}{6GS^2} + \frac{4}{3K} \tag{14}$$

where $6GS^2$ is the effective compressive stiffness found in the previous section for a circular slice due to the steel–rubber bond.

The *ad hoc* modification recommended by Gent and Meinecke (1970) and later adopted by Stanton and Roeder (1982) provides the effective compressive modulus by

$$\frac{1}{E_1} = \frac{1}{E_0(1+2\beta S^2)} + \frac{1}{K} \tag{15}$$

where the unaltered compression modulus E_0 is not neglected compared to the supplementary stiffness provided by the bond. However, E_0 is generally small compared to $2E_0\beta S^2$ and for high shape factor the term $1/K$ controls the value of E_1. Equation (14) approximates the exact formula given by the Bessel functions solution better than eqn (15).

For large arguments, the Bessel functions are expressed by their asymptotic expansions

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{1}{8x} + \frac{9}{128x^2} + \cdots\right]$$

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{3}{8x} - \frac{15}{128x^2} - \cdots\right].$$

Using the bionomial formula and accounting for the first few terms only we have

$$I_1(x)/I_0(x) = 1 - 1/2x - 1/8x^2$$

and the following expression for the effective compression stiffness is obtained

$$E_1 = K\left[1 - \frac{1}{(12G/K)^{1/2}S} + \frac{1}{(48G/K)S^2}\right]. \tag{16}$$

Equation (16) can be used with negligible error when the shape factor is greater than 24. It is also interesting to note that the above formulas when applied in the appropriate range of the shape factor can be used to determine the bulk compression coefficient K.

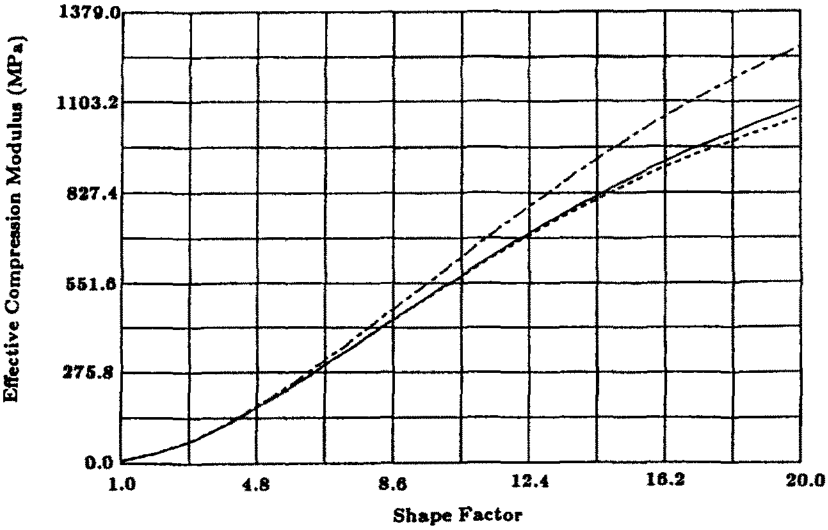

Fig. 3. Effective compression modulus vs shape factor. Exact solution, eqn (11), ———; recommended formula, eqn (14), -------; *ad hoc* modified expression, (15), — — — —; $G = 1.4$ MPa, $K = 2070$ MPa.

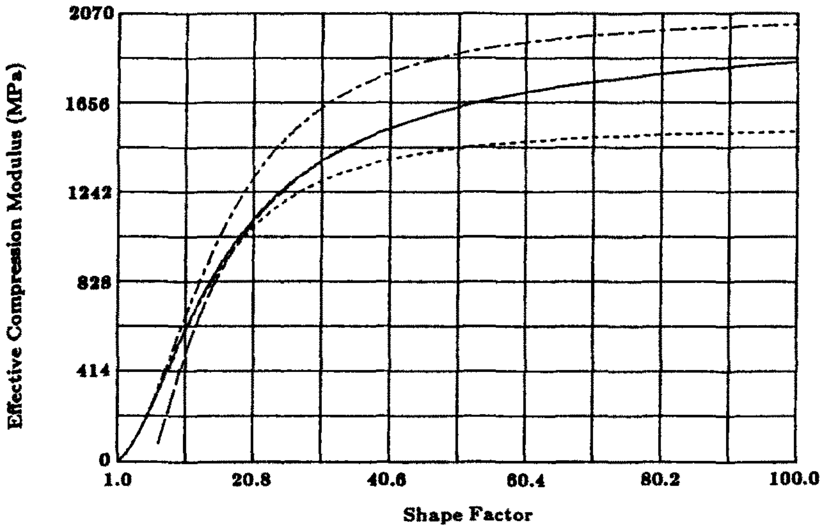

Fig. 4. Effective compression modulus vs shape factor. Exact solution, eqn (11), ———; recommended for $1 < S < 24$, eqn (14), ------; recommended for $24 < S$, eqn (16), — — —; *ad hoc* modified formula, eqn (15), — — —; $G = 1.4$ MPa, $K = 2070$ MPa.

Equations (11), (14), (15) and (16) were plotted for comparison (Figs 3 and 4). The *ad hoc* modification formula over-estimates the value of E_1. Instead, the other formulas should be used for design or for the estimation of K through experiments determining E_1 or G.

FLEXURE EXCLUDING VOLUME CHANGE

When a circular slice is subjected to a tilt of angle α about the y-axis, eqn (3) is solved to determine the tilting stiffness

$$p_{,rr} + \frac{1}{r} p_{,r} + \frac{1}{r^2} p_{,\theta\theta} - \frac{12G\alpha}{t^3} r \sin\theta = 0 \tag{17}$$

with the condition of zero pressure on the edge. The angle θ is measured from the y-axis

(Fig. 2). A solution of the form $p(r)\sin\theta$ is considered because it is null on the y-axis and reaches maximum amplitude on the x-axis. Replacing in eqn (17), it transforms to

$$p_{,rr}+\frac{1}{r}p_{,r}-\frac{1}{r^2}p-\frac{12G\alpha}{t^3}\cdot r=0.$$

The function $(3G\alpha/2t^3)r^3$ is a particular solution of the above equation. A homogeneous solution of the form r^n leads to the indicial equation $n=\pm 1$, and the general solution has the form

$$p(r)=\frac{3G\alpha}{2t^3}r^3+C_6r+C_7/r$$

where C_6 and C_7 are arbitrary constants. Since the pressure is expected to be bounded over the pad, $C_7\equiv 0$, and the condition $p(R)=0$ yields $C_6=-(3G\alpha R^2/2t^3)$, and finally

$$p(r,\theta)=\frac{3G\alpha}{2t^3}(r^3-R^2r)\sin\theta. \tag{18}$$

Note that the vertical resultant $\int_A p(r,\theta)\,\mathrm{d}A$ is zero. The tilting stiffness is obtained from the expression for the moment

$$M=\int_A p(r,\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta=\frac{G\alpha}{8t^3}\pi R^6 \tag{19}$$

and since α/t is the curvature $1/\rho$, $\pi R^4/4$ is the moment of inertia I, and $R^2/4t^2=S^2$, we have $M=2GS^2I/\rho=(EI)_1^\infty/\rho$ and thus $(EI)_1^\infty=2GS^2I$. Using the result obtained for the compressive stiffness when the effect of compressibility was excluded, namely $E_1^\infty=6GS^2$ the above result is interpreted as if the effective moment of inertia is reduced by a factor of 3 (three). Chalhoub and Kelly (1988) concluded that the effective moment of inertia is reduced by a factor of 5 (five) for long rectangular bearings.

FLEXURE INCLUDING VOLUME CHANGE

When the effect of bulk compression is included, eqn (4) is solved for the pressure distribution over a circular pad subjected to pure flexure

$$p_{,rr}+\frac{1}{r}p_{,r}+\frac{1}{r^2}p_{,\theta\theta}-\lambda^2\left(p+\frac{Kr}{\rho}\sin\theta\right)=0 \tag{20}$$

with the condition of zero pressure on the edge. A particular solution of eqn (20) is $-Kr\sin\theta/\rho$. The homogeneous equation corresponding to (21) can be transformed by separation of variables. Letting $p(r,\theta)=\Omega(r)T(\theta)$, and substituting in the homogeneous equation, we have, $T(\theta)=C_8\cos n\theta+C_9\sin n\theta$, where C_8 and C_9 are arbitrary constants and n is an integer. Applying the boundary condition of zero pressure on the y-axis, $p(r,0)=0$ and $p(r,\pi)=0$, yields $T(\theta)=C_9\sin n\theta$. Without loss of generality consider $n=1$. The differential equation in Ω, has the form $r^2\Omega_{,rr}+r\Omega_{,r}-(\lambda^2r^2+1)\Omega=0$ which is a Bessel equation of order one and parameter λ. It has a solution of the form $\Omega(r)=C_{10}I_1(\lambda r)+C_{11}K_1(\lambda r)$, where I_1 and K_1 are the modified Bessel functions of first and second kind respectively, and of order one. C_{10} and C_{11} are arbitrary constants. Since a bounded solution is expected, $C_{11}\equiv 0$. The remaining boundary condition of zero pressure on the delimiting circle, $p(R,\theta)=0$, provides C_{10} and thus

$$p(r,\theta) = \frac{\alpha K}{t}\left[R\frac{I_1(\lambda r)}{I_1(\lambda R)} - r\right]\sin\theta. \tag{21}$$

Equation (21) represents the pressure distribution on the face of the pad. The moment resisted under this pressure distribution is obtained by integration over the area

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^R p(r,\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{\pi K\alpha}{t}\frac{R^2}{\lambda^2}(\lambda R I_2(\lambda R)/I_1(\lambda R) - (\lambda R)^2/4). \end{aligned}$$

Expressing $I_2(x)$ for small arguments as the sum of its first few terms

$$I_2(x) = \frac{x^2}{8} + \frac{x^4}{96} + \frac{x^6}{3072} + \cdots.$$

Setting the ratio $I_2(x)/I_1(x)$, using the binomial formula and neglecting the terms of order higher than 6, leads to

$$I_2(\lambda R)/I_1(\lambda R) = \lambda R/4 - (\lambda R)^3/96 + (\lambda R)^5/1536.$$

The expression for M becomes

$$\begin{aligned} M &= -\frac{\pi K\alpha}{t}\frac{\lambda^2 R^6}{96}(1-\lambda^2R^2/1536) \\ &= -GS^2(1-3GS^2/K)I/\rho = -(EI)_1/\rho, \end{aligned}$$

and the effective bending stiffness is then

$$(EI)_1 = GIS^2(1-3GS^2/K). \tag{22}$$

The term outside the brackets is the tilting stiffness for the incompressibility case found in the previous section, here reduced by a factor of $3GS^2/K$ because bulk compression was taken into account. Notice also that this reduction factor is lower than the one found for cross sections in the shape of long rectangles in Chalhoub and Kelly (1988). When compressibility was considered, the latter was $15GS^2/4K = 3.75GS^2/K$. Equation (22) has the limitation that it can not be used beyond a certain value of S since it was obtained by considering the Bessel functions for $S \leqslant (K/12G)^{1/2}$.

The equations for the tilting stiffnesses with and without the effect of bulk compressibility were derived to illustrate how their values are over-estimated when the elastomer is considered incompressible. For conciseness, no discussions and comparisons will be pursued here for the tilting stiffness of a circular base isolator. Instead, the theoretical values for the compressive stiffness obtained from the expressions developed in the previous sections will be compared to some experimental values.

BRIEF DESCRIPTION OF A RECENT EXPERIMENT

At the Earthquake Engineering Research Center, tests were performed on half-scaled circular bearings. The bearings were similar to ones already in use under the first base-isolated building in the U.S., the Foothills Communities Law and Justice Center in Rancho Cucamonga, described by Kelly (1986).

The bearings that were tested consisted of several rubber slices interleaved by steel plates. Each bearing had 30 layers of DURO 62 natural rubber, 0.61 cm thick each, 29 steel

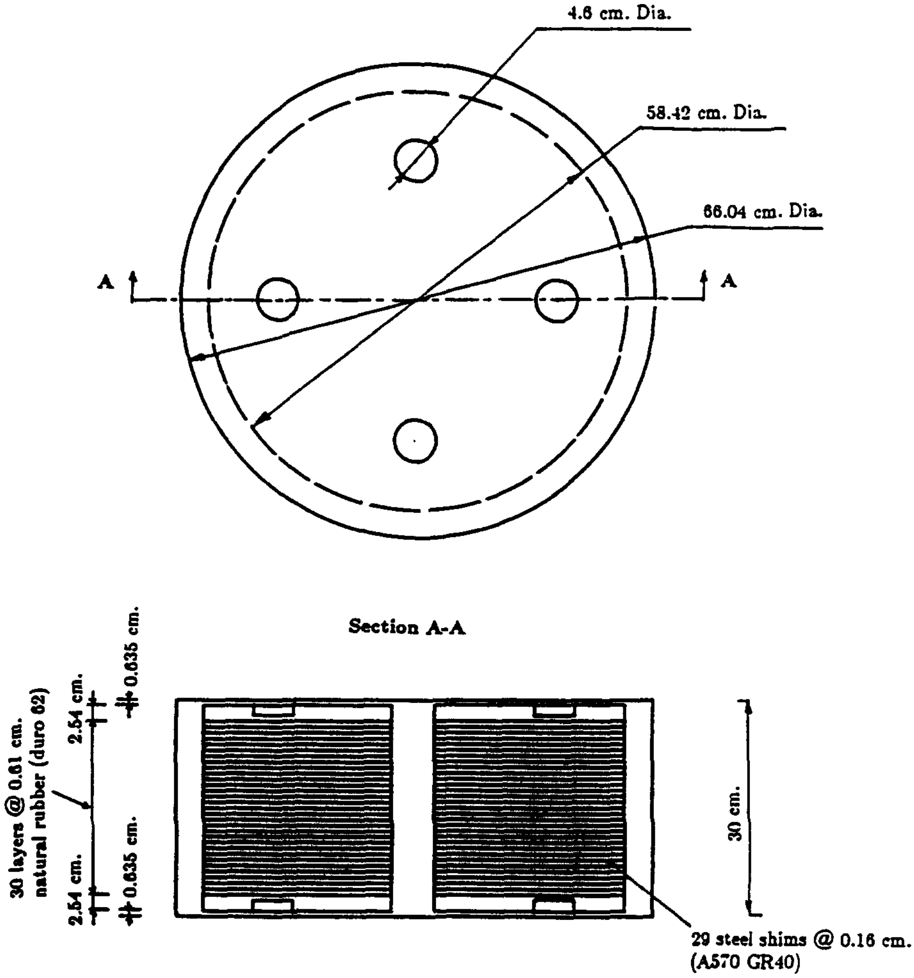

Fig. 5. Multilayered rubber bearing showing some dimensions.

shims 0.16 cm thick each, an upper and lower end plates 2.5 cm thick each with four 4.6 cm in diameter dowels designed to accommodate steel pegs. In real usage, the pegs transmit the shear force from the base beams of the structure to the bearings. The steel and rubber are bonded together under high temperature and high pressure. The effect of the bond is a substantial increase in the vertical stiffness of the assemblage, while the shear (horizontal) stiffness is affected to a much lesser extent. The high vertical stiffness avoids the amplification of rocking and the high horizontal flexibility causes the structure to move like a single degree of freedom oscillator at low frequency. This type of movement protects the structure from the amplification of ground-borne accelerations. A bearing is shown in Fig. 5.

The bearings were tested by two pairs on a rig that had two vertical and one horizontal actuators. When the bearing is sheared in the horizontal direction, it tends to deflect vertically. The vertical actuators were force controlled in the sense that they were set to apply a fixed vertical load while providing the necessary vertical movement in order to keep this load constant when the bearing deflects vertically. The vertical load capacity of the rig was about 6500 kN. The horizontal actuator was interactively displacement controlled in the sense that it was pre-set to apply a certain displacement signal while developing the necessary horizontal force needed to shear the bearings correspondingly. The horizontal actuator had a capacity of about 1780 kN. The actuator could produce a cyclic horizontal movement with an amplitude of about ± 23 cm. The test apparatus is shown in Fig. 6.

The test program consisted of horizontal and vertical cycling of the bearings to measure their compressive and shear stiffnesses respectively. The vertical load was consecutively fixed to values ranging from 310–1870 kN. In the following, one result from shear tests will

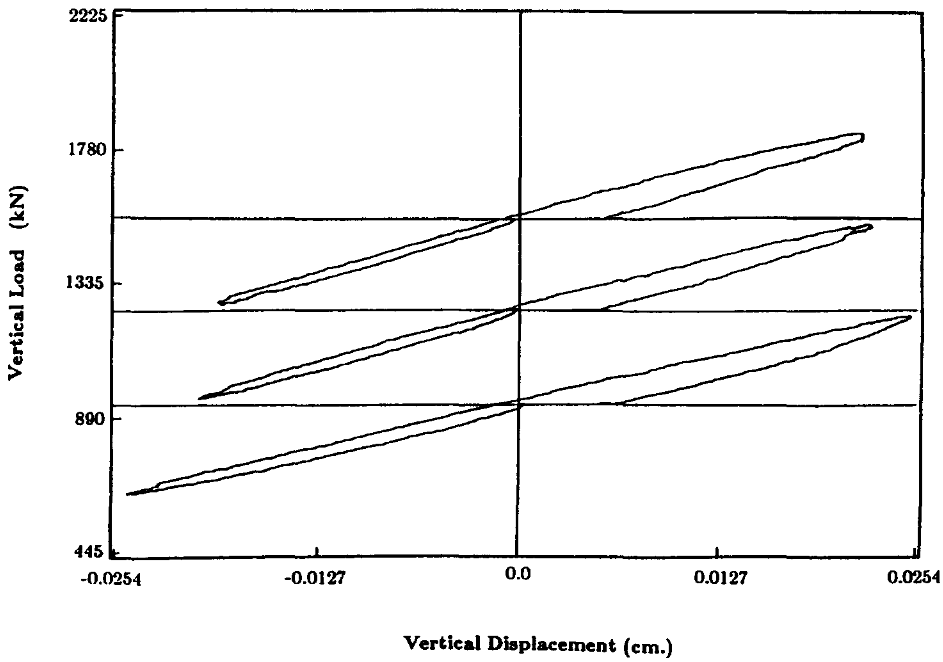

Fig. 7. Vertical cycling test—axial load vs axial displacement.

be used to determine the shear modulus G, of the elastomer. The rest of the section will be devoted to a comparison between the experimental vertical stiffness of the bearing and the ones obtained from the formulas developed in the previous sections of this paper.

CORRELATION BETWEEN THEORY AND EXPERIMENT

It was found that the vertical stiffness increased with vertical load and reached a constant value for loads equal and larger than 670 kN. A number of tests consisted of loading the bearing to a certain level of vertical load and then applying a vertical cyclic force of ± 220 kN. The slope of the load-deflection curve yielded the vertical stiffness of those bearings at vertical loads of 934 ± 220 kN, 1245 ± 220 kN, and 1557 ± 220 kN. The measured vertical stiffness ranged between 12.259 MN/cm and 15.324 MN/cm. More precisely, the vertical stiffness K_v was of 12.259 MN/cm for an average vertical load P_{ave} of 934 kN, $K_v = 13.572$ MN/cm for $P_{ave} = 1245$ kN, and $K_v = 15.324$ MN/cm for $P_{ave} = 1557$ kN. These results are shown in Fig. 7. It is assumed that the steel–rubber bond and the rubber compressibility enter into effect when the bearing is loaded vertically by a force in the range of or larger than the service load. For this reason the value $K_v = 15.324$ MN/cm will be compared to the ones obtained from the theoretical solution.

First G was obtained from a shear test at a shear strain of about $\gamma = 50\%$. Since there was almost no bending when the bearings were subjected to this level of shear strain, the value for G was estimated fairly closely. For $\gamma = 50\%$, the horizontal force per bearing was of 267 kN. The total height of a bearing being 30 cm, the horizontal displacement of an end plate with respect to the other was of 15 cm. The horizontal stiffness was then

$$K_H = 267/15 = 17.8 \text{ kN/cm}$$

and the shear modulus was obtained from

$$K_H = GA_0/t_h$$

where A_0 is the cross sectional area of the pad, and t_h is the total thickness of the rubber.

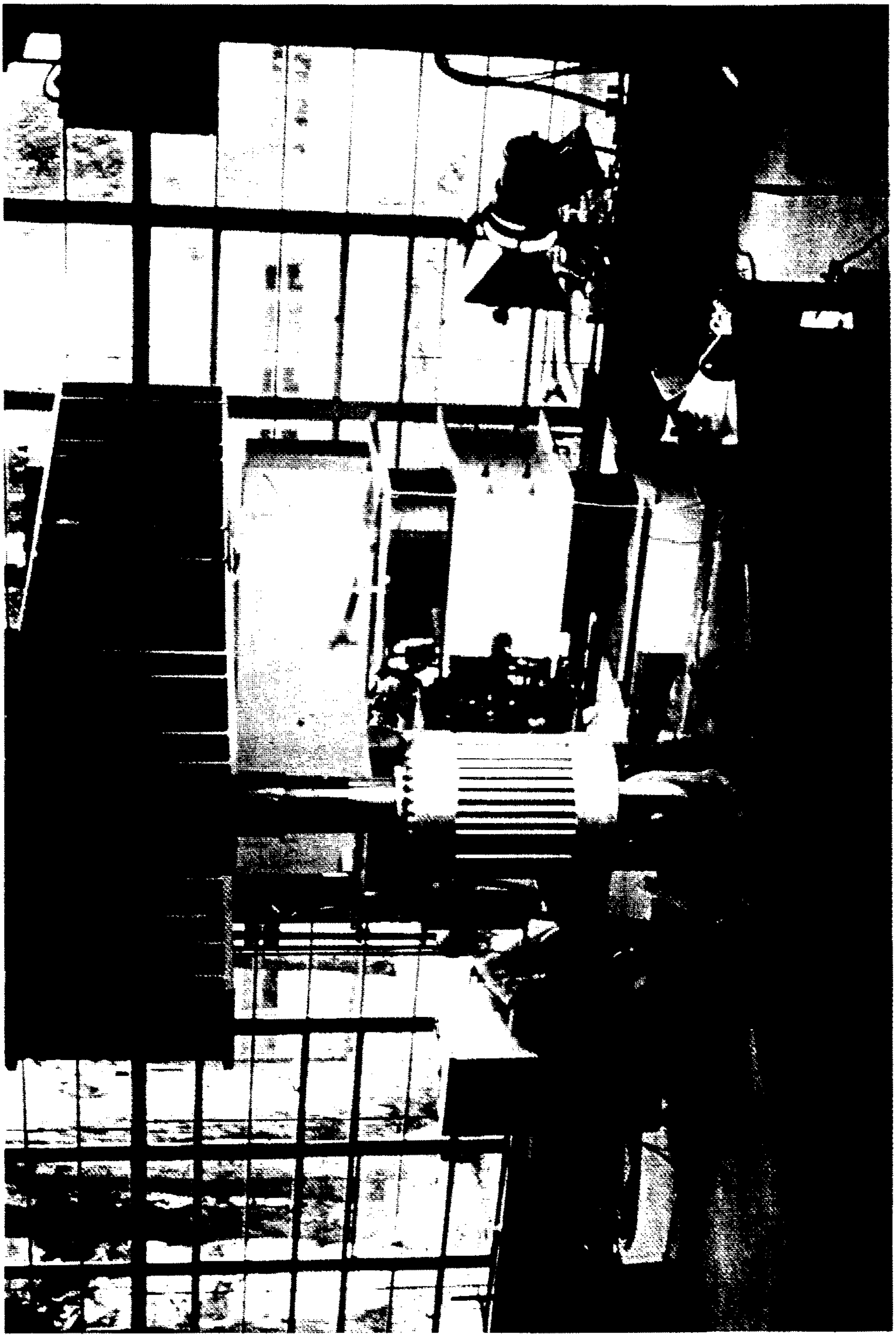

Fig. 6. Test apparatus. Two pairs of rubber bearings in compression.

For the 30 layers at a thickness of 0.61 cm each, $t_h = 18.3$ cm. The bearing had a radius of $R_0 = 33$ cm and thus $A_0 = 3421.2$ cm^2. Thus the equation above gives $G = 0.952$ MPa.

In computing the effect of the shape factor we use the area of the confined rubber which corresponds to the radius of the steel plates $R = 29.21$ cm. The shape factor is then $S = R/2t = 23.93$, and the effective compressive stiffness excluding bulk compressibility becomes $E_1^\infty = 6GS^2 = 3270$ MPa. The corresponding vertical stiffness of the pad is

$$K_v = E_1^\infty A_c/t_h.$$

With $E_1^\infty = 3270$ MPa, $A = 2680.48$ cm^2, and $t_h = 18.3$ cm the vertical stiffness is $K_v = 47.897$ MN/cm. This value over-estimates the measured stiffness by a factor of about 3 (three).

Consider the semi-empirical *ad hoc* formula presented by Allen *et al.* (1966) and by Stanton and Roeder (1982)

$$\frac{1}{E_1} = \frac{1}{E_1^\infty} + \frac{1}{K}$$

where K is the bulk modulus. For natural rubber DURO 62, K is about 2070 MPa and the above equation yields an effective compression modulus $E_1 = 1267.584$ MPa and a vertical stiffness $K_v = 18.567$ MN/cm. This value still over-estimates the measured vertical stiffness by 21%.

Now consider eqn (14) developed here and recommended for shape factors up to around 25 after it was compared to the exact solution in eqn (11) (see also Figs 3 and 4)

$$\frac{1}{E_1} = \frac{1}{6GS^2} + \frac{4}{3K}$$

it yields $E_1 = 1052.706$ MPa and $K_v = 15.419$ MN/cm. The ratio of this value to the experimentally measured one is of 1.006, showing excellent agreement.

Equation (16) which is recommended in this paper for high shape factor ($S \geqslant 24$)

$$E_1 = K\left[1 - \frac{1}{(12G/K)^{1/2}S} + \frac{1}{(48G/K)S^2}\right]$$

yields $E_1 = 1069.344$ MPa and $K_v = 15.663$ MN/cm. The ratio of this value to the measured one is of 1.02, also showing good agreement. Note that this equation becomes very accurate when S increases and that it was initially recommended for S greater or equal to 24.

In order to make an accurate comparison, no safety factors were used in the preceding discussion. The formulas recommended in this paper agreed extremely well with the measured value. The classically used formulas where one ignores bulk compressibility and the other accounts for it by an *ad hoc* modification, yielded an over-estimated stiffness.

CONCLUSIONS

With the acceptance of base isolation, there is an increasing need to study the behavior of multilayered rubber bearings. Pads made of rubber slices interleaved by steel plates are basic components in any practical base isolation system. Rubber and steel are vulcanized together to ensure bond and several design formulas were proposed to account for this effect. However, a major handicap in the existing formulas is that they mainly consider rubber as incompressible. In this paper, the equations governing the pressure in the rubber were developed by taking into account bulk compressibility.

Compression tests were recently performed on cylindrical rubber pads and experimental results were compared to the ones predicted by the proposed equations.

It is concluded that:

- The additional stiffness E_{Γ}^{c} caused by the kinematic constraint (steel–rubber bond) is proportional to the square of the shape factor S. Even for low shape factors of values around 10, and for $G = 0.9$ MPa, the additional stiffness due to the kinematic constraint is much higher than the Young's modulus of rubber. The latter one can be neglected in design expressions.
- When bulk compressibility of the rubber compound is taken into account, E_{Γ}^{c} is reduced by a quantity which is proportional to the fourth power of the shape factor S. This reduction becomes more important when rubber layers are thinner.
- The exact solution involves Bessel functions that were expanded to yield two simple expressions, for $S \leqslant 24$ and $S \geqslant 24$, respectively. When these expressions are plotted for S varying from 0 to 100 they match exactly the exact solution for their respective ranges of application. When compared to test results, both recommended formulas [eqns (14) and (16)] gave very accurate results for a pad of shape factor equal to 24.
- The classically used *ad hoc* modified formula that accounts for bulk compressibility over-estimated the measured stiffness by 20%, and this over-estimation increases with higher shape factors. Such a difference is unacceptable especially because it is on the unconservative side.

In summary, the present work shows the importance of bulk compressibility of rubber in bonded blocks and provides two simplified design formulas that were checked against experimental data and proven accurate.

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APPENDIX

Considering an undeformed pad (Fig. 1a) sandwiched between two rigid steel plates, it is assumed that, under direct compression horizontal planes remain plane and horizontal and that a vertical line deforms into a parabola. The displacements are assumed to have the form:

$$\begin{aligned} u(x,y,z) &= u_0(x,y)(1-4z^2/t^2) \\ v(x,y,z) &= v_0(x,y)(1-4z^2/t^2) \\ w(x,y,z) &= w(z) \end{aligned} \tag{A1}$$

the strains are then expressed as

$$\begin{aligned} \varepsilon_{xx} &= u_{0,x}(1-4z^2/t^2) \\ \varepsilon_{yy} &= v_{0,y}(1-4z^2/t^2) \\ \varepsilon_{zz} &= w_{,z} \\ \gamma_{xy} &= (u_{0,y}+v_{0,x})(1-4z^2/t^2) \\ \gamma_{yz} &= -8zv_0/t^2 \\ \gamma_{zx} &= -8zu_0/t^2. \end{aligned} \tag{A2}$$

The preceding equations represent the kinematic assumptions. The volume change in an element of the material is given by

$$\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} = e \tag{A3}$$

or

$$(u_{0,x}+v_{0,y})(1-4z^2/t^2)+w_{,z}-e = 0.$$

Integration of the above equation over an element of unit area and through the thickness of the pad leads to

$$\tfrac{2}{3}(u_{0,x}+v_{0,y})+[w(t/2)-w(-t/2)]-et = 0$$

from which

$$\tfrac{2}{3}(u_{0,x}+v_{0,y})-\varepsilon_c-e = 0. \tag{A4}$$

The first two equations of equilibrium, in absence of body forces for a thin rubber layer, are

$$\begin{aligned}\sigma_{xx,x}+\sigma_{xz,z} &= 0\\ \sigma_{yy,y}+\sigma_{yz,z} &= 0\end{aligned} \tag{A5}$$

where the shear stress σ_{xy} is neglected compared to the normal stresses σ_{ii} and the shear stress in the vertical planes (x,z) and (y,z). Furthermore, considering the normal stress as being equal in all directions to the hydrostatic pressure in the rubber, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, we have from eqn (A5)

$$\sigma_{xz,z} = p_{,x}, \quad \sigma_{yz,z} = p_{,y}. \tag{A6}$$

Within the range of strains considered, the behavior of the material is assumed linear elastic, satisfying Hooke's Law and the shear stress–strain relationship is

$$\sigma_{xy} = G\gamma_{xy}, \quad \sigma_{yz} = G\gamma_{yz}, \quad \sigma_{zx} = G\gamma_{zx}. \tag{A7}$$

Since $\gamma_{xz} = -8zu_0/t^2$ and $\gamma_{yz} = -8zv_0/t^2$ from eqns (A2), and differentiating the expressions in eqn (A6) with respect to x and y respectively, we have

$$p_{,xx}+p_{,yy} = -\frac{8G}{t^2}(u_{0,x}+v_{0,y})$$

but $(u_{0,x}+v_{0,y}) = \frac{3}{2}(\varepsilon_c+e)$ from eqn (A4), thus

$$p_{,xx}+p_{,yy} = -\frac{12G}{t^2}(\varepsilon_c+e). \tag{A8}$$

If incompressibility is assumed, $e = 0$ and eqn (A8) becomes

$$\nabla^2 p+12G\varepsilon_c/t^2 = 0. \tag{A9}$$

If the material is considered compressible with a bulk modulus K, then the volume change is $e = -p/K$ and eqn (A8) becomes

$$\nabla^2 p-\lambda^2(p-K\varepsilon_c) = 0 \tag{A10}$$

where $\lambda^2 = 12G/Kt^2$.

In the case of pure flexure of the same pad considered earlier (Fig. 1c), the displacements are assumed to have the form

$$\begin{aligned}u(x,y,z) &= u_0(x,y)(1-4z^2/t^2)-z^2\alpha/2t\\ v(x,y,z) &= v_0(x,y)(1-4z^2/t^2)\\ w(x,y,z) &= z\alpha x/t.\end{aligned} \tag{A11}$$

The rest of the derivation of the equations in the pressure follows the same procedure used for the compression case. It led to

$$\tfrac{2}{3}(u_{0,x}+v_{0,y})+\alpha x/t-e = 0 \tag{A12}$$

instead of eqn (A4). The equation in the pressure is then

$$\nabla^2 p - \frac{12G\alpha}{t^3} x = 0 \tag{A13}$$

for the incompressibility case, and

$$\nabla^2 p - \lambda^2 \left(p + \frac{K\alpha}{t} x \right) = 0 \tag{A14}$$

for the case where the bulk compression modulus K is considered finite.